Lecture 19 — 04/01/2019

Electromagnetic Plane Waves (Cont'd)

Wave Packets

By superposing plane waves with different frequencies, one can obtain wave packets that last for a finite time and have finite spatial extension. Ignoring polarization of the wave, a wave packet can be written in the following form (assuming one-dimensional space):

$$u(x,t) = \int A(k)\, e^{i(kx-\omega t)}\, dk$$

For non-dispersive media, $k = \frac{\omega}{c} n$, where n is constant. Then:

$$u(x,t) = \int A(k)\, e^{ik(x - \frac{c}{n}t)}\, dk = u(x - vt, 0) \qquad \left(v = \frac{c}{n}\right)$$

In this case, the wave packet is just translated in space without any change in its shape.

For dispersive media, $n = n(\omega)$ and $k(\omega) = \frac{\omega}{c} n(\omega)$. Assuming that $A(k)$ is significant over a small range of k around k_0, we have:

$$\omega(k) = \omega(k_0) + \omega'(k_0)(k - k_0) + \frac{\omega''(k_0)}{2}(k - k_0)^2 + \dots$$

Then:

$$u(x,t) = \int A(k)\, e^{i[kx - \omega(k_0)t - \omega'(k_0)(k-k_0)t + \dots]}\, dk \Rightarrow u(x,t) \simeq$$

$$e^{i[k_0 x - \omega(k_0)t]} \int A(k)\, e^{i(kx - v_g t)}\, dk$$

Here $v_g \equiv \omega'(k_0)$ is called the "group velocity". Therefore:

$$u(x,t) \simeq e^{i(k_0 x - \omega_0 t)}\, u(x - v_g t, 0) \qquad (\omega_0 \equiv \omega(k_0))$$

This is just a plane wave modulated by a wave packet that moves at speed v_g without distortion.

Next, let us consider the second-order term $\frac{\omega''(k_0)}{2}(k - k_0)^2$. Then:

$$u(x,t) \simeq e^{i(k_0 x - \omega_0 t)} \int A(k)\, e^{ik(x - v_g t)}\, e^{-\frac{i}{2}(k-k_0)^2 \omega''(k_0)}\, dk$$

The last term in the integral leads to <u>spreading</u> of the wave packet. To see this in more detail, we consider the following example.

Example: Wave packet with a Gaussian profile.

$$A(k) = e^{-\frac{\sigma^2}{2}(k - k_0)^2}$$

Thus:

$$u(x,t) = e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{+\infty} e^{-\frac{\sigma^2}{2}(k-k_0)^2 - \frac{i}{2}(k-k_0)^2 \omega''(k_0) t + i(k-k_0)(x - v_g t)} \, dk$$

The integral becomes:

$$\int_{-\infty}^{+\infty} e^{-\left(\frac{\sigma^2}{2} + \frac{i}{2}\omega''(k_0) t\right)(k-k_0)^2 + i(k-k_0)(x - v_g t)} \, dk$$

By completing the square in the exponent, we find:

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\sigma^2 + i\omega''(k_0) t)(k-k_0)^2 + i(k-k_0)(x - v_g t)} \, dk = \sqrt{\frac{2\pi}{\sigma^2 + i\omega''(k_0) t}}$$

$$\exp\left[-\frac{(x - v_g t)^2}{\sigma^2 + i\omega'' t}\right] = \sqrt{\frac{2\pi}{\sigma^2 + i\omega''(k_0) t}} \quad \exp\left[-\frac{(x - v_g t)^2}{\sigma^4 + \omega''^2_{(k_0)} t^2} (\sigma^2 - i\omega''(k_0) t)\right]$$

$$= \sqrt{\frac{2\pi}{\sigma^2 + i\omega''(k_0) t}} \quad \exp\left[-\frac{(x - v_g t)^2}{\sigma^2 + \frac{\omega''^2(k_0) t^2}{\sigma^2}}\right] \quad \exp\left[\frac{i}{2} \frac{\omega''(k_0)(x - v_g t)^2 t}{\sigma^4 + \omega''^2_{(k_0)} t^2}\right]$$

Hence:

$$|u(x,t)| \sim \frac{(2\pi)^{1/2}}{(\sigma^4 + \omega''^2_{(k_0)} t^2)^{1/4}} \quad \exp\left[-\frac{(x - v_g t)^2}{\sigma^2 + \frac{\omega''^2_{(k_0)} t^2}{\sigma^2}}\right]$$

The width of the wave packet at time t is:

$$\sigma(t) = \sqrt{\sigma^2 + \frac{\omega''^2(k_0)\, t^2}{\sigma^2}}$$

At this time the center of the wave packet has traveled a distance $v_g t$. We see that the narrower the wave packet is, i.e., the larger σ is, the faster it spreads. It takes a time $\sim \frac{\sigma^2}{\omega''(k_0)}$ for the wave packet to spread to $\sqrt{2}$ of its original width. The quantity $\omega''(k_0)$ is thus called the "group-velocity dispersion".

One can show that the envelope of $u(x,t)$ with the quadratic dispersion approximation obey a Schrödinger-like equation in the variables t and $X \equiv x - v_g t$.